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Thermal constriction resistance with convective boundary conditions—2. Layered half-space contacts

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Abstract—This paper is concerned with the axisymmetric thermal analysis of a contact on a semi-infinite single layer which is perfectly attached to a half-space. The Hankel integral transform method is employed and convective boundary conditions are imposed on the contact surface. In each case suitable Fourier expansions reduce the problem to the solution of integro-differential equations similar to those studied in Part 1 for half-space contacts. Compact expressions are developed and the variation of thermal constriction resistance is shown in non-dimensional form for a wide range of Biot numbers, layer thickness and thermal conductivity ratios.

1. INTRODUCTION

IN PART 1 of this work, the authors examined the variation of thermal constriction resistance for circular contacts on a half-space, with convective (Robin) boundary conditions. If we attach a finite thickness layer to this half-space model, we obtain a *coated* surface, which may be used for enhancing thermal contact conductance. This was studied experimentally in refs. [1, 2], with applications in the microelectronics industry.

Analytical studies were conducted in ref. [3] using Hankel transform theory, however, these were restricted to isoflux and *approximately isothermal* contacts, both problems having an insulated external boundary. In particular, Negus *et al.* [3] did not solve the actual mixed boundary-value problem but used a linear superposition of known flux distributions to circumvent mathematical difficulties.

The aim of this work is to examine the thermal behaviour of layered half-space contacts with mixed convective boundary conditions. The integral Hankel transform approach will be used as outlined in Part 1. The thermal constriction resistance [4] of these contacts is shown to vary with a dimensionless reference Biot number, as well as a wide range of layer thickness and conductivity ratios. In all cases studied, suitable Fourier expansions permit efficient and accurate solutions of the resulting integro-differential equations, in a likewise manner that was studied in Part 1.

2. PROBLEM STATEMENT

From Fig. 1, we note that the two temperature fields in regions 1 (layer) and 2 (substrate) are joined at $\zeta = \delta$, where, for perfect contact, the boundary conditions are

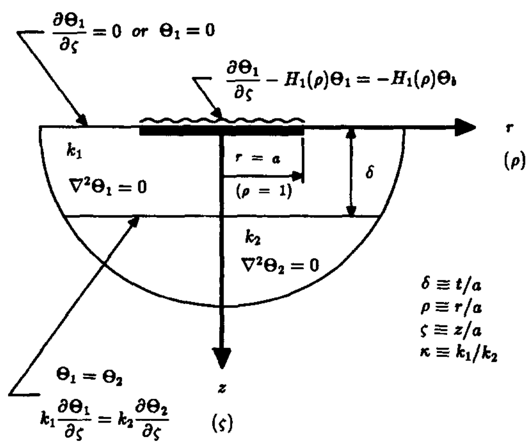

FIG. 1. Coated half-space model.

$$\Theta_1 = \Theta_2 \quad (1)$$

$$\kappa\frac{\partial\Theta_1}{\partial\zeta} = \frac{\partial\Theta_2}{\partial\zeta}. \quad (2)$$

The term κ is the thermal conductivity ratio of the two materials defined by $\kappa \equiv k_1/k_2$.

The temperature fields in the layer and substrate will be denoted by the non-dimensional Hankel forms

$$\Theta_1(\rho,\zeta) = \mathscr{H}_0[\xi^{-1}\{A(\xi)\exp(-\xi\zeta) + B(\xi)\exp(\xi\zeta)\}; \rho] \quad (3)$$

$$\Theta_2(\rho,\zeta) = \mathscr{H}_0[\xi^{-1}D(\xi)\exp(-\xi\zeta); \rho]. \quad (4)$$

Thus, from boundary conditions (1) and (2), we can obtain

$$A(\xi) = \beta D(\xi) \quad (5)$$

NOMENCLATURE

a	contact radius dimension
a_n	Fourier series expansion coefficient and solution vector
A	standard system matrix
$A(\xi)$	Hankel transformed temperature function
b_n	Fourier series expansion coefficient
B	coefficient matrix in external convection problems
$B(\xi)$	layer transformed temperature function
c	non-uniform flux and convection distribution parameter
c_n	Fourier series expansion coefficient
$C(\xi)$	Hankel layer transformed temperature function
d	non-uniform flux and convection distribution parameter
d_m	Legendre series expansion coefficient
D	symmetric coefficient matrix
$D(\xi)$	layer transformed temperature function
e_m	Legendre series expansion coefficient
$E(\lambda)$	complete elliptic integral of the second kind
$f(x)$	Fourier transform function
f_{m+n}	Fourier series coefficient
F	symmetric coefficient matrix
$F(\rho)$	external integral equation function of ρ
g_n	Legendre series expansion coefficient and vector
$G(\rho)$	external integral equation function of ρ
h, h_1, h_2	convection coefficients
h_{m+n}	Fourier series coefficient
H_i, H_1, H_2	dimensionless Biot numbers, $h_i a/k$
I	identity matrix
$j(x)$	Fourier transform function
k, k_1, k_2	thermal conductivity, layer and substrate conductivities
$K(\lambda)$	complete elliptic integral of the first kind
$K'(\lambda)$	complementary complete elliptic integral of the first kind
m, n	integer constants
n_T	truncation value of system of equations
N	diagonal coefficient matrix
P_n	Legendre polynomial of degree n
$q(\rho), q_0(\rho), q_0$	heat flux functions, uniform heat flux
Q, Q^*	total heat flux through contact, total flux, Q/ak
r	radial coordinate
$r_{m,n}$	layer influence matrix entries
R	influence matrix for layer problems
R_c	thermal constriction resistance
t	thickness of layer in positive z-direction
T, T_∞	domain temperature, reference temperature
U	Heaviside unit step function
w	Legendre polynomial argument, $2\rho^2-1$
x	Fourier transformed coordinate
z	depth coordinate.

Greek symbols

α	thermal conductivity parameter, $(1-\kappa)/(1+\kappa)$
β	thermal conductivity parameter, $(1+\kappa)/2\kappa$
β_n	Fourier series expansion coefficient
γ	parameter in Lipschitz–Hankel integrals
δ	relative thickness, t/a
$\delta_{n,0}$	delta function, equals unity only for $n = 0$
ε	modulus parameter in theta-functions
ζ	dimensionless depth coordinate, z/a
$\Theta, \bar{\Theta}_c, \Theta_0, \Theta_b$	temperature excess, mean contact temperature excess, specified base and contact temperature excesses
Θ_1, Θ_2	layer temperature, substrate temperature
θ	angular coordinate
κ	conductivity ratio, k_1/k_2
λ	modulus for elliptic integrals
μ, ν	general integers
ξ	transformed radial coordinate
π	constant, 3.14159265...
ρ	dimensionless radial coordinate, r/a
ϕ	angular coordinate
Ψ_c	dimensionless constriction factor, $4ak_1R_c$.

Other symbols

$\mathscr{A}_1, \mathscr{A}_2$	Abel integral operator transforms
∂	partial derivative operation
$\mathscr{F}_c, \mathscr{F}_s$	Fourier cosine and sine transform operators
$\mathscr{H}_\nu$	Hankel transform of order ν
∇	Laplacian operator in cylindrical polar coordinates.

$$B(\xi) = -\alpha\beta \exp(-2\delta\xi) D(\xi) \quad (6)$$

where

$$\alpha = \frac{1-\kappa}{1+\kappa}, \quad \beta = \frac{1+\kappa}{2\kappa}. \quad (7)$$

Similarly as in Part 1, we will define the thermal constriction resistance of the circular contact on the layered half-space to be

$$R_c = \frac{\bar{\Theta}_{1c}}{Q} \quad (8)$$

where the mean contact temperature rise $\bar{\Theta}_{1c}$, and total heat flux over the contact, Q, are defined in dimensional and non-dimensional coordinates as

$$\bar{\Theta}_{1c} = \frac{1}{\pi a^2}\int_0^a \Theta_1(r,0)2\pi r\,\mathrm{d}r \tag{9}$$

$$= 2\int_0^1 \Theta_1(\rho,0)\rho\,\mathrm{d}\rho \tag{10}$$

$$Q = -\int_0^a \frac{\partial\Theta_1(r,0)}{\partial z}2\pi r\,\mathrm{d}r \tag{11}$$

$$= -2\pi k_1 a\int_0^1 \frac{\partial\Theta_1(\rho,0)}{\partial\zeta}\rho\,\mathrm{d}\rho. \tag{12}$$

Again we will present results in terms of the dimensionless thermal constriction resistance factor

$$\Psi_c = 4ak_1R_c. \tag{13}$$

We will consider two general problem types with mixed convective boundary conditions.

Case (A): *contact conductance on the internal* ($\rho < 1$) *region with external* ($\rho > 1$) *insulation*

$$\frac{\partial\Theta}{\partial\zeta} - H_1(\rho)\Theta = -H_1(\rho)\Theta_b, \quad \rho < 1 \tag{14}$$

$$\frac{\partial\Theta}{\partial\zeta} = 0, \quad \rho > 1. \tag{15}$$

Case (B): *contact conductance on the internal* ($\rho < 1$) *region with external* ($\rho > 1$) $T = 0$ *condition*

$$\frac{\partial\Theta}{\partial\zeta} - H_1(\rho)\Theta = -H_1(\rho)\Theta_b, \quad \rho < 1 \tag{16}$$

$$\Theta = 0, \quad \rho > 1. \tag{17}$$

Using forms (3)–(6), we will be able to solve the mixed surface boundary conditions in terms of having only one unknown transformed temperature, $D(\xi)$, in the integral solution. The Hankel transforms used are as outlined in Part 1 of this paper. Additionally we will be concerned here with the layer ratios δ and κ.

3. LIMITING CASES

Limiting solutions will also be required here to verify the upper and lower bounds for the convective boundary conditions in equations (14)–(17). These are similar to the forms given in Part 1, except that they will require the necessary approximations to the integral equations, and thus are generally more involved than the half-space models only. Three cases are discussed here, not including varying-flux contacts, which could be similarly developed as was done for the half-space in Part 1.

Case (i)

$$\frac{\partial\Theta_1}{\partial\zeta} = -q_0, \quad \rho < 1 \tag{18}$$

$$\frac{\partial\Theta_1}{\partial\zeta} = 0, \quad \rho > 1. \tag{19}$$

Surface conditions are *unmixed*, and therefore the solution procedure is straightforward, that is

$$B(\xi) - A(\xi) = q_0\frac{J_1(\xi)}{\xi}. \tag{20}$$

Using equations (5) and (6), and incorporating the binomial theorem to convert the quotient into a convergent series, after some manipulation we may show that

$$A(\xi)+B(\xi) = q_0\frac{J_1(\xi)}{\xi} \times\left[1+2\sum_{\mu=1}^{\infty}(-1)^\mu\alpha^\mu\exp(-2\mu\delta\xi)\right] \tag{21}$$

and hence write

$$\Theta_1(\rho,0) = q_0\left[\int_0^\infty \frac{J_1(\xi)}{\xi}J_0(\xi\rho)\,\mathrm{d}\xi + 2\sum_{\mu=1}^{\infty}(-1)^\mu\alpha^\mu\int_0^\infty\frac{\mathrm{e}^{-2\mu\delta\xi}}{\zeta}J_1(\xi)J_0(\xi\rho)\,\mathrm{d}\xi\right]. \tag{22}$$

The mean contact temperature $\bar{\Theta}_{1c}$ and total heat flux Q are now easily evaluated, and we may obtain the dimensionless constriction factor based on the thermal conductivity k_1 of the layer

$$\Psi_c = \frac{32}{3\pi^2}+\frac{16}{\pi}\sum_{\mu=1}^{\infty}(-1)^\mu\alpha^\mu\int_0^\infty\frac{\mathrm{e}^{-2\mu\delta\xi}}{\xi^2}J_1^2(\xi)\,\mathrm{d}\xi. \tag{23}$$

This result was also obtained by Negus *et al.* [3], who approximated the infinite integral for a specified range $\mu\delta \geqslant 0.5$, but evaluated it numerically for $\mu\delta < 0.5$. Fortunately, we note that this is a Lipschitz–Hankel type integral, which may be *explicitly* reduced solely in terms of complete elliptic integrals, as discussed by ref. [5]. A summary of these is given in ref. [6], and we may obtain

$$\Psi_c = \frac{32}{3\pi^2}+\frac{16}{\pi}\sum_{\mu=1}^{\infty}(-1)^\mu\alpha^\mu\left[\frac{4}{3\pi}(1+(\mu\delta)^2)^{1/2} \times\{(\mu\delta)^2(K(\lambda)-E(\lambda))+E(\lambda)\}-\mu\delta\right] \tag{24}$$

where the modulus λ, is defined as

$$\lambda = (1+(\mu\delta)^2)^{-1/2}. \tag{25}$$

Case (ii)

$$\frac{\partial\Theta_1}{\partial\zeta} = -q_0, \quad \rho < 1 \tag{26}$$

$$\Theta_1 = 0, \quad \rho > 1. \tag{27}$$

Using equations (5) and (6), and letting

$$C(\xi) = D(\xi)(1-\alpha\,\mathrm{e}^{-2\delta\xi}) \tag{28}$$

our Hankel-form boundary conditions become

$$\mathscr{H}_0[G(\xi)C(\xi);\rho] = \frac{q_0}{\beta}, \quad \rho < 1 \tag{29}$$

$$\mathscr{H}_0[\xi^{-1}C(\xi);\rho] = 0, \quad \rho > 1. \tag{30}$$

The term $G(\xi)$ is defined by

$$G(\xi) = \frac{1+\alpha\,\mathrm{e}^{-2\delta\xi}}{1-\alpha\,\mathrm{e}^{-2\delta\xi}} \tag{31}$$

$$= 1+2\sum_{\mu=1}^{\infty}\alpha^{\mu}\,\mathrm{e}^{-2\mu\delta\xi} \tag{32}$$

$$= 1+g(\xi). \tag{33}$$

Due to condition (30), we may employ the Abel integral operator $\mathscr{A}_2^{-1}$ to obtain

$$\mathscr{F}_s[C(\xi);x] = (\tfrac{1}{2}\pi)^{1/2}f(x)U(x-1). \tag{34}$$

As in Part 1, we next use two *simultaneous Fourier* expansions for the unknown variable $f(x)$

$$f(x) = F(\theta) = \sum_{n=0}^{\infty} a_n\cos(2n+1)\theta, \quad 0<\theta<\pi/2 \tag{35}$$

$$\sin\theta F(\theta) = \sum_{n=0}^{\infty} b_n\cos(2n+1)\theta, \quad 0<\theta<\pi/2 \tag{36}$$

and the representations $x = \cos\theta$, $\rho = \cos\phi$. Now with these expansions, we can note that

$$C(\xi) = \int_0^1 f(x)\sin\xi x\,\mathrm{d}x \tag{37}$$

$$= \sum_{n=0}^{\infty} a_n\int_0^{\pi/2}\cos(2n+1)\theta\sin(\xi\cos\theta)\sin\theta\,\mathrm{d}\theta \tag{38}$$

$$= \frac{\pi}{2}\sum_{n=0}^{\infty} b_n(-1)^n J_{2n+1}(\xi). \tag{39}$$

The method now follows similar lines as described for the half-space analysis, that is, to reduce the system to a *single* integro-differential equation. We choose the Fourier sine transform of the $g(\xi)C(\xi)$ function in this case to be represented by

$$\mathscr{F}_s[g(\xi)C(\xi);x] = \left(\frac{\pi}{2}\right)^{1/2}\sum_{n=0}^{\infty} c_n\cos(2n+1)\theta \tag{40}$$

$$= \left(\frac{\pi}{2}\right)^{1/2} j(x). \tag{41}$$

The c_n are now related to the b_n (for easiest analytical manipulation), by

$$c_n = \sum_{m=0}^{\infty} r_{m,n}b_m \tag{42}$$

where, we can show that

$$r_{m,n} = \frac{2}{\pi}(-1)^m\int_0^{\infty} g(\xi)J_{2m+1}(\xi) \times \int_0^{\pi}\sin(\xi\cos\phi)\cos(2n+1)\phi\,\mathrm{d}\phi\,\mathrm{d}\xi \tag{43}$$

$$= 4(-1)^{m+n}\sum_{\mu=1}^{\infty}\alpha^{\mu} \times \int_0^{\infty}\mathrm{e}^{-2\mu\delta\xi}J_{2m+1}(\xi)J_{2n+1}(\xi)\,\mathrm{d}\xi. \tag{44}$$

After further manipulation with equation (29), we may obtain the infinite system of linear equations, in matrix form

$$\{[I]+[R][D]\}a_n = g_n \tag{45}$$

where now, the right-hand side vector becomes

$$g_0 = \frac{2}{\pi\beta}q_0;\quad g_n = 0,\ n = 1,2,\ldots \tag{46}$$

Now, with the form of equation (39), we note that with equation (28), the temperature solution on the top surface becomes

$$\Theta_1(\rho,0) = \beta\mathscr{H}_0[\xi^{-1}C(\xi);\rho] \tag{47}$$

$$= \frac{\pi}{2}\beta\sum_{n=0}^{\infty} b_n(-1)^n\int_0^{\infty} J_{2n+1}(\xi)J_0(\xi\rho)\,\mathrm{d}\xi. \tag{48}$$

After further integrations, for a uniform flux q_0, we may obtain

$$\bar{\Theta}_{1c} = \frac{\pi}{2}\beta b_0,\quad \Psi_c = 2\frac{\beta b_0}{q_0}. \tag{49}$$

Results for this case are given in Tables 1 and 2.

Table 1. Isoflux contact with $T = 0$ external; $\delta < 1$

Conductivity ratio, $\kappa = k_1/k_2$	Thickness ratio, $\delta = t/a$	Constriction factor, $\Psi_c = 4ak_1R_c$
0.01	0.01	2.0452×10^{-2}
0.10	0.01	7.4119×10^{-2}
0.50	0.01	2.8797×10^{-1}
0.75	0.01	4.1574×10^{-1}
1.25	0.01	6.6225×10^{-1}
2.00	0.01	1.0132×10^{0}
10.00	0.01	3.8699×10^{0}
100.00	0.01	1.2122×10^{1}
0.01	0.10	1.2471×10^{-1}
0.10	0.10	1.8277×10^{-1}
0.50	0.10	3.6852×10^{-1}
0.75	0.10	4.6028×10^{-1}
1.25	0.10	6.1132×10^{-1}
2.00	0.10	7.8389×10^{-1}
10.00	0.10	1.4322×10^{0}
100.00	0.10	1.8300×10^{0}
0.01	0.50	3.9713×10^{-1}
0.10	0.50	4.2417×10^{-1}
0.50	0.50	4.9453×10^{-1}
0.75	0.50	5.2081×10^{-1}
1.25	0.50	5.5558×10^{-1}
2.00	0.50	5.8608×10^{-1}
10.00	0.50	6.5381×10^{-1}
100.00	0.50	6.7702×10^{-1}

Table 2. Isoflux contact with $T = 0$ external; $\delta \geqslant 1$

Conductivity ratio, $\kappa = k_1/k_2$	Relative thickness, $\delta = t/a$	Constriction factor, $\Psi_c = 4ak_1R_c$
0.01	1.00	4.9863×10^{-1}
0.10	1.00	5.0702×10^{-1}
0.50	1.00	5.2785×10^{-1}
0.75	1.00	5.3513×10^{-1}
1.25	1.00	5.4435×10^{-1}
2.00	1.00	5.5205×10^{-1}
10.00	1.00	5.6794×10^{-1}
100.00	1.00	5.7303×10^{-1}
0.01	2.00	5.3304×10^{-1}
0.10	2.00	5.3452×10^{-1}
0.50	2.00	5.3820×10^{-1}
0.75	2.00	5.3947×10^{-1}
1.25	2.00	5.4106×10^{-1}
2.00	2.00	5.4238×10^{-1}
10.00	2.00	5.4505×10^{-1}
100.00	2.00	5.4590×10^{-1}
0.01	10.00	5.4031×10^{-1}
0.10	10.00	5.4033×10^{-1}
0.50	10.00	5.4036×10^{-1}
0.75	10.00	5.4037×10^{-1}
1.25	10.00	5.4039×10^{-1}
2.00	10.00	5.4040×10^{-1}
10.00	10.00	5.4042×10^{-1}
100.00	10.00	5.4043×10^{-1}
0.01	100.00	5.4038×10^{-1}
100.00	100.00	5.4038×10^{-1}

Case (iii)

$$\Theta_1 = \Theta_0, \quad \rho < 1 \tag{50}$$

$$\frac{\partial \Theta_1}{\partial \zeta} = 0, \quad \rho > 1. \tag{51}$$

By defining here

$$C(\xi) = D(\xi)(1 + \alpha\, e^{-2\delta\xi}) \tag{52}$$

then in Hankel form, equations (50) and (51) become

$$\mathscr{H}_0[\xi^{-1}G(\xi)C(\xi); \rho] = \frac{\Theta_0}{\beta}, \quad \rho < 1 \tag{53}$$

$$\mathscr{H}_0[C(\xi); \rho] = 0, \quad \rho > 1. \tag{54}$$

The $G(\xi)$ is in this case given by

$$G(\xi) = \frac{1 - \alpha\, e^{-2\delta\xi}}{1 + \alpha\, e^{-2\delta\xi}} \tag{55}$$

$$= 1 + 2 \sum_{\mu=1}^{\infty} (-1)^{\mu} \alpha^{\mu} e^{-2\mu\delta\xi} \tag{56}$$

$$= 1 + g(\xi). \tag{57}$$

This time, we employ the representation

$$\mathscr{F}_c[A(\xi); x] = (\tfrac{1}{2}\pi)^{1/2} f(x) U(1-x) \tag{58}$$

where U is the Heaviside unit function, defined by

$$U(1-x) = 1, \quad x < 1$$
$$= 0, \quad x > 1 \tag{59}$$

and the expansions

$$f(x) = F(\theta) = \sum_{n=0}^{\infty} a_n \sin(2n+1)\theta, \quad 0 < \theta < \pi/2 \tag{60}$$

$$\sin\theta F(\theta) = \sum_{n=0}^{\infty} b_n \sin(2n+1)\theta, \quad 0 < \theta < \pi/2. \tag{61}$$

With these simultaneous Fourier expansions for $f(x)$, we may show that

$$C(\xi) = \frac{\pi}{2} \sum_{n=0}^{\infty} a_n (2n+1)(-1)^n \frac{J_{2n+1}(\xi)}{\xi} \tag{62}$$

$$\mathscr{F}_c[g(\xi)C(\xi); x] = \left(\frac{\pi}{2}\right)^{1/2} \sum_{n=0}^{\infty} a_n (2n+1)(-1)^n \times \int_0^{\infty} \frac{J_{2n+1}(\xi)}{\xi} g(\xi) \cos \xi x \, d\xi \tag{63}$$

$$= \left(\frac{\pi}{2}\right)^{1/2} j(x). \tag{64}$$

Now, with $x = \cos\theta$, as before, we use the representation

$$\sin\theta J(\theta) = \sum_{n=0}^{\infty} c_n \sin(2n+1)\theta \tag{65}$$

and then the integro-differential equation will reduce to the system of equations

$$b_n + c_n = g_n. \tag{66}$$

The b_n are related to the a_n by the $d_{m,n}$ in the Appendix (Part 1), and the c_n are related to the a_n by the symmetric $r_{m,n}$ given by

$$r_{m,n} = \frac{2}{\pi}(2m+1)(-1)^m \int_0^{\infty} g(\xi) \frac{J_{2m+1}(\xi)}{\xi} \times \int_0^{\pi} \cos(\xi \cos\phi) \sin(2n+1)\phi \sin\phi \, d\phi \, d\xi \tag{67}$$

$$= 4(2m+1)(2n+1)(-1)^{m+n} \sum_{\mu=1}^{\infty} \alpha^{\mu}(-1)^{\mu} \times \int_0^{\infty} \frac{e^{-2\mu\delta\xi}}{\xi^2} J_{2m+1}(\xi) J_{2n+1}(\xi) \, d\xi. \tag{68}$$

The matrix system hence becomes

$$\{[D] + [R]\}a = g \tag{69}$$

and the g_n are as defined for Case (ii) by replacing q_0 with Θ_0. To evaluate the heat flux, we note that

$$\frac{\partial \Theta}{\partial \zeta} = -\beta \mathscr{H}_0[C(\xi); \rho] \tag{70}$$

$$= -\beta \sum_{n=0}^{\infty} a_n (2n+1)(-1)^n \int_0^{\infty} J_{2n+1}(\xi) J_0(\xi\rho) \, d\xi. \tag{71}$$

To find Q, we use equation (12), and thus obtain

$$Q = \frac{\pi^2}{2}\beta k a_0, \quad \Psi_c = \frac{8\Theta_0}{\beta\pi^2 a_0}. \tag{72}$$

Again we included the effects of the thermal conductivity ratio κ into constants α and β. We note from equation (7), that $\alpha \to 1$ as κ becomes very small. In terms of material conductivities, as the substrate conductivity k_2 becomes very large compared to the layer conductivity k_1, the substrate behaves like a *thermal sink*, with $\Theta = 0$ on $\zeta = \delta$.

Negus *et al.* [3] approximated condition (50) by superposition of two flux distributions; the uniform flux in Case (i), and an *equivalent isothermal* flux, that is, the flux resulting from an isothermal contact on a *half-space* (Part 1), the solution of which is straightforward. Comparison is made, between the exact solution to the mixed boundary value problem derived here and the solution from ref. [3], in Table 3.

4. CONTACT CONDUCTANCE, CASE (A)

The boundary conditions for this problem are as stated in equations (14) and (15), and with the forms defined in equations (3) and (4), we may write our Hankel-form boundary conditions as

$$\mathscr{H}_0[C(\xi);\rho]+H_1(\rho)\mathscr{H}_0[\xi^{-1}G(\xi)C(\xi);\rho] = -H_1(\rho)\frac{\Theta_b}{\beta}, \quad \rho<1 \tag{73}$$

$$\mathscr{H}_0[C(\xi);\rho]=0, \quad \rho>1. \tag{74}$$

In the above, $C(\xi)$ is defined by equation (52) and $G(\xi)$ is given by equation (55). Next, we use equation (58) to reduce to a single integro-differential equation along with expansions (60) and (61). Now with

$$g(\xi) = 2\sum_{\mu=1}^{\infty}(-1)^{\mu}\alpha^{\mu}\,e^{-2\mu\delta\xi} \tag{75}$$

then we have for $\rho < 1$

$$\mathscr{H}_0[C(\xi);\rho]+H_1(\rho)\mathscr{H}_0[\xi^{-1}C(\xi);\rho]$$
$$+H_1(\rho)\mathscr{H}_0[\xi^{-1}g(\xi)C(\xi);\rho] = -H_1(\rho)\frac{\Theta_b}{\beta}. \tag{76}$$

The function $C(\xi)$ may be given by equation (62), and thus, with the forms of equations (63)–(65) we can obtain similarly a system of equations, for uniform H_1

$$a_n + H_1(2n+1)^{-1}(b_n+c_n) = g_n \tag{77}$$

and in matrix notation

$$\{[I]+H_1[N]([D]+[R])\}a_n = g_n. \tag{78}$$

The c_n are related to the a_n through equation (42) by the $r_{m,n}$ given by equation (68). The b_n are related to the a_n as in equation (66), and the g_n are simply given by

Table 3. Isothermal contact with external insulation; verifications

$\delta = t/a$	$\kappa = k_1/k_2$	$\Psi_c/4$ Negus *et al.* [3]	$\Psi_c/4$ equation (75)	Percentage difference (%)
0.01	0.01	5.830×10^{-3}	5.821×10^{-3}	−0.15
0.01	0.10	2.875×10^{-2}	2.893×10^{-2}	0.62
0.01	0.50	1.279×10^{-1}	1.284×10^{-1}	0.37
0.01	2.00	4.893×10^{-1}	4.873×10^{-1}	−0.41
0.01	10.00	2.220×10^{0}	2.179×10^{0}	−1.90
0.01	100.00	1.346×10^{1}	1.161×10^{1}	−15.90
0.10	0.01	3.206×10^{-2}	3.180×10^{-2}	−0.83
0.10	0.10	5.436×10^{-2}	5.422×10^{-2}	−0.26
0.10	0.50	1.463×10^{-1}	1.465×10^{-1}	0.10
0.10	2.00	4.327×10^{-1}	4.323×10^{-1}	−0.09
0.10	10.00	1.368×10^{0}	1.361×10^{0}	−0.05
0.10	100.00	4.109×10^{0}	4.095×10^{0}	−0.36
1.00	0.01	1.581×10^{-1}	1.587×10^{-1}	0.36
1.00	0.10	1.692×10^{-1}	1.698×10^{-1}	0.34
1.00	0.50	2.105×10^{-1}	2.112×10^{-1}	0.31
1.00	2.00	3.076×10^{-1}	3.085×10^{-1}	0.29
1.00	10.00	5.021×10^{-1}	5.032×10^{-1}	0.22
1.00	100.00	8.497×10^{-1}	8.510×10^{-1}	0.16
10.00	0.01	2.392×10^{-1}	2.421×10^{-1}	1.20
10.00	0.10	2.405×10^{-1}	2.435×10^{-1}	1.22
10.00	0.50	2.454×10^{-1}	2.484×10^{-1}	1.20
10.00	2.00	2.564×10^{-1}	2.594×10^{-1}	1.16
10.00	10.00	2.771×10^{-1}	2.801×10^{-1}	1.06
10.00	100.00	3.123×10^{-1}	3.154×10^{-1}	0.97
100.00	0.01	2.489×10^{-1}	2.519×10^{-1}	1.18
100.00	100.00	2.562×10^{-1}	2.592×10^{-1}	1.16

$$g_0 = \frac{2}{\pi}\frac{H_1\Theta_b}{\beta}; \quad g_n = 0, n = 1, 2, \ldots \tag{79}$$

If we had a *non-uniform* symmetric contact conductance as defined in equation (36) of Part 1, then in a similar manner, our matrix system would become

$$\left\{[I]+H_1\left(\frac{c}{2}+d\right)[N]([D]+[R]) + H_1\frac{c}{2}[N][G]([D']+[R'])\right\}a = g. \tag{80}$$

The $[D']$ and $[R']$ are again *rectangularized* versions of $[D]$ and $[R]$ as discussed in Part 1. The g_n in this case would be as given in Part 1, and dividing each by the factor β. For either a uniform or non-uniform contact conductance, evaluation of the mean contact temperature and total heat flux remains the same. Thus we proceed

$$\Theta(\rho,0) = \beta\mathscr{H}_0[\xi^{-1}G(\xi)C(\xi);\rho] \tag{81}$$

$$= \beta\{\mathscr{H}_0[\xi^{-1}C(\xi);\rho] + \mathscr{H}_0[\xi^{-1}g(\xi)C(\xi);\rho]\}. \tag{82}$$

We note that we may also represent $C(\xi)$ in terms of b_n by

$$C(\xi) = \sum_{n=0}^{\infty} b_n \int_0^{\pi/2} \sin(2n+1)\theta\cos(\xi\cos\theta)\,\mathrm{d}\theta \tag{83}$$

and thus

$$\mathscr{H}_0[\xi^{-1}C(\xi);\rho] = \sum_{n=0}^{\infty} b_n \int_0^{\infty} J_0(\xi\rho) \times \int_0^{\pi/2} \sin(2n+1)\theta\cos(\xi\cos\theta)\,\mathrm{d}\theta\,\mathrm{d}\xi. \tag{84}$$

To evaluate $\bar{\Theta}_{1c}$ this form is easily reducible, and thus

$$\bar{\Theta}_{1c} = \beta\left[\frac{\pi}{2}b_0 + \frac{\pi}{2}\sum_{n=0}^{\infty} 4(-1)^{m+1}(2m+1)(2n+1) \times \sum_{\mu=1}^{\infty}(-1)^{\mu}\alpha^{\mu}\int_0^{\infty}\frac{\mathrm{e}^{-2\mu\delta\xi}}{\xi^2}J_{2n+1}(\xi)J_1(\xi)\,\mathrm{d}\xi\right] \tag{85}$$

$$\bar{\Theta}_{1c} = \frac{\beta\pi}{2}\left[b_0 + \sum_{n=0}^{\infty} a_n r_{n,0}\right] \tag{86}$$

where $r_{n,0}$ are the first row entries of equation (68) in matrix $[R]$, which are evaluated beforehand. The total heat flux Q will again be given by equation (12), and thus we obtain the simple forms

$$Q = \frac{\pi^2}{2}ak\beta a_0, \quad \Psi_c = \frac{4}{\pi}\left[\frac{b_0 + \sum_{n=0}^{\infty} a_n r_{n,0}}{a_0}\right]. \tag{87}$$

We note that when the contact conductance is *uniform*, then as in Part 1, we can express the solution for the mean contact temperature in terms of the much simpler expression for the total heat flux, equation (87). Thus we note from Part 1, for *uniform* contact conductance only

$$\bar{\Theta}_{1c} = \Theta_b - \frac{Q^*}{H_1\pi}, \quad \Psi_c = 4\frac{\bar{\Theta}_{1c}}{Q^*} = \frac{4}{\pi}\left(\frac{2\Theta_b}{\pi\beta a_0} - \frac{1}{H_1}\right) \tag{88}$$

where $Q^* \equiv Q/ak$.

5. CONTACT CONDUCTANCE, CASE (B)

Boundary conditions (16) and (17), may be cast in Hankel form as

$$\mathscr{H}_0[G(\xi)C(\xi);\rho] + H_1(\rho)\mathscr{H}_0[\xi^{-1}C(\xi);\rho] = -H_1(\rho)\frac{\Theta_b}{\beta}, \quad \rho < 1 \tag{89}$$

$$\mathscr{H}_0[\xi^{-1}C(\xi);\rho] = 0, \quad \rho > 1 \tag{90}$$

where $C(\xi)$ is denoted by equation (28), and $G(\xi)$ by equation (31). We next choose representation (34) and expansions (35) and (36), and analogously we find our system of equations, for a *uniform* conductance coefficient, to be

$$\{[I]+[R][D]+H_1[N][D]\}a_n = g_n. \tag{91}$$

The $d_{m,n}$, $r_{m,n}$ entries are respectively given by equation (49) of Part 1, and (43), and the g_n by equation (79). For a *non-uniform symmetric* conductance coefficient of a form similar to equation (36) of Part 1, the system of equations we obtain are

$$\left\{[I]+[R][D]+H_1\left(\frac{c}{2}+d\right)[N][D] + H_1\frac{c}{2}[N][G][D']\right\}a_n = g_n. \tag{92}$$

The entries of $[G]$ are again as defined for equation (80) (see also Part 1), with $[D']$ again being the rectangularized version of $[D]$, and the g_n would again be given as in the previous section. For either a uniform or non-uniform contact conductance, we obtain for the mean contact temperature and total heat flux

$$\bar{\Theta}_{1c} = \frac{\pi}{2}\beta b_0, \quad Q = \frac{\pi^2}{2}\beta ak\left[a_0 + \sum_{n=0}^{\infty} b_n r_{n,0}\right]. \tag{93}$$

Again, the $r_{n,0}$ are the first row entries defined by equation (43). Thus, our expression for the dimen-

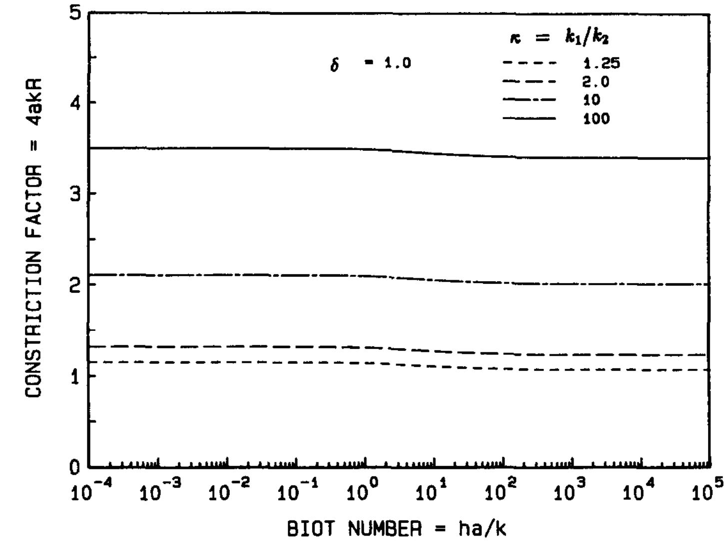

FIG. 2. Constriction factor vs Biot number; $\kappa > 1$, $\delta = 1.0$ external insulated boundary.

sionless constriction factor becomes after some manipulation

$$\Psi_c = \frac{4}{\pi} \frac{b_0}{a_0 + \sum_{n=0}^{\infty} b_n r_{n,0}}. \tag{94}$$

Again, for a *uniform* contact conductance, we may use the form of equation (88) for simplification, and thus obtain

$$\Psi_c = 4\frac{\bar{\Theta}_{1c}}{Q^*} = \frac{2\beta b_0}{H_1\left(\Theta_b - \frac{\pi}{2}\beta b_0\right)}. \tag{95}$$

6. PRESENTATION OF RESULTS

An explicit expression (24), was derived for Case (i), overcoming the previous resort [3] to numerical integration. Tables 1 and 2 provide the variation of constriction factor with relative thickness and conductivity ratios, for an isoflux contact with $T = 0$ external boundary. Thus, along with the results shown in ref. [3], these respectively determine the upper and lower bounds for an isoflux contact with external *convection* boundary conditions on a layered half-space. In Table 3, results are tabulated for an isothermal contact with external insulation. Constriction factors ($\Psi_c/4$) are compared to those tabulated in ref. [3]. As mentioned in ref. [3], the superposition technique provided *approximately* isothermal contact conditions, but remarkably as seen in Table 3, the percentage error between the two techniques was usually less than 1%.

Extensive tabulated results for uniform contact conductance with external insulation or $T = 0$ boundary conditions can be found [1]. The models developed

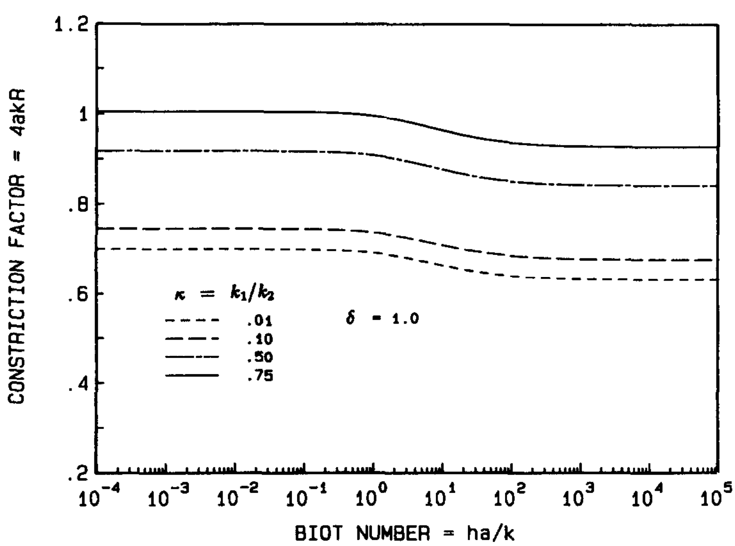

FIG. 3. Constriction factor vs Biot number; $\kappa < 1$, $\delta = 1.0$ external insulated boundary.

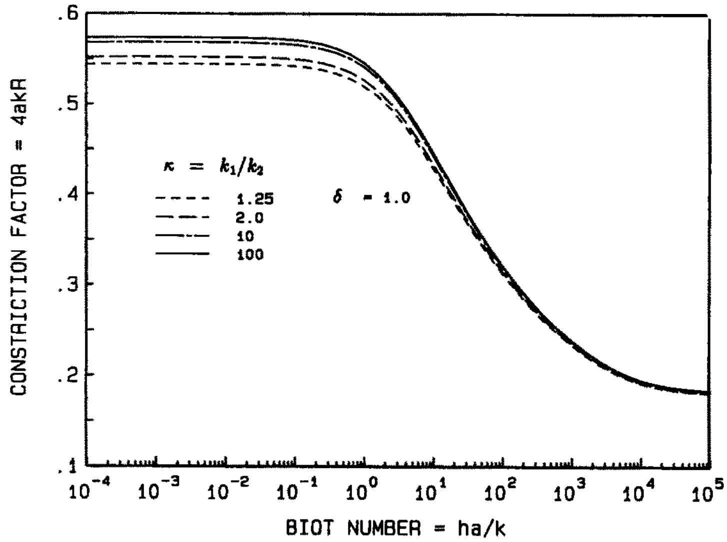

FIG. 4. Constriction factor vs Biot number; $\kappa > 1$, $\delta = 1.0$ external $T = 0$ boundary.

also easily allow for non-uniform contact conductance, however, these are too extensive to be presented in the context of this paper. Similar behaviour would be observed as in the non-uniform convection studies conducted on the half-space in Part 1. Figures 2 and 3 illustrate results for contacts with a *uniform* contact conductance, and an insulated external boundary. In Fig. 2, the conductivity ratios are greater than 1.0, and a relative thickness $\delta = 1.0$ was chosen for example. This represents a coating (layer) that is *conductive* relative to the substrate material. In all cases, the constriction resistance *increases* with *increasing* conductivity ratio κ, and with *decreasing* relative thickness δ. Also, along with these trends, the percentage difference between the upper and lower bounds on the solution *increases* steadily, to a maximum of 10% for $\delta = 0.1$, $\kappa = 100$, as noted in ref. [6]. Figure 3 gives results for conductivity ratios *less* than 1.0 ($\delta = 1.0$). For these cases, the coating is termed *resistive* compared to the substrate. Here we observe that the constriction resistance *decreases* with *decreasing* relative thickness δ, and *decreasing* conductivity ratio. The decreasing conductivity ratio essentially provides for an improving *heat sink* at $\zeta = \delta$.

Results are shown in Figs. 4–7 for a uniform contact conductance with external $T = 0$ boundary. Figures 4 and 5 consider conductivity ratios greater than 1.0, and Figs. 6 and 7 show results for conductivity ratios less than 1.0. We note that for conductivity ratios *greater* than 1.0, the constriction resistance *increases* with *decreasing* relative thickness, and *increasing* conductivity ratios. For conductivity ratios *less* than 1.0 (*a resistive layer*), the constriction resistance *decreases* with *decreasing* conductivity ratio, and *decreasing* relative thickness. Different combinations of conductivity ratio κ and relative thickness δ, yield a *simi-*

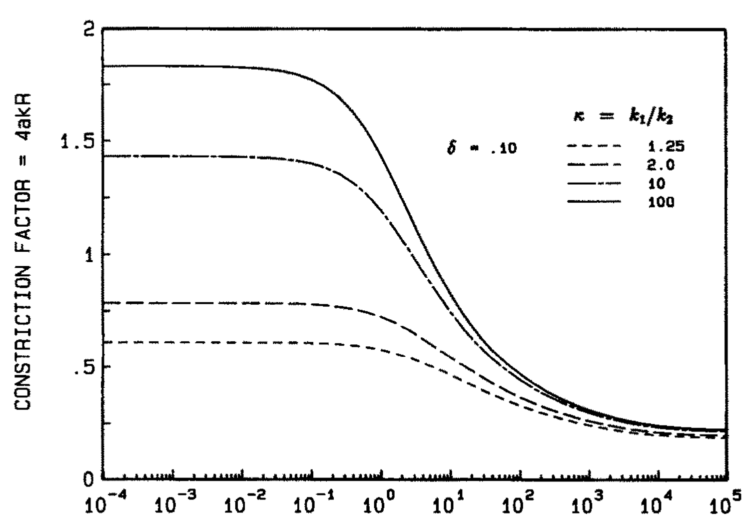

FIG. 5. Constriction factor vs Biot number; $\kappa > 1$, $\delta = 0.1$ external $T = 0$ boundary.

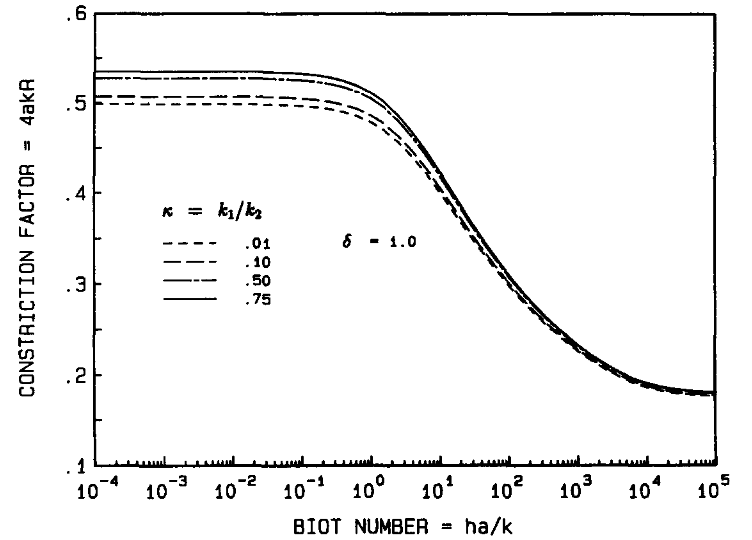

FIG. 6. Constriction factor vs Biot number; $\kappa < 1$, $\delta = 1.0$ external $T = 0$ boundary.

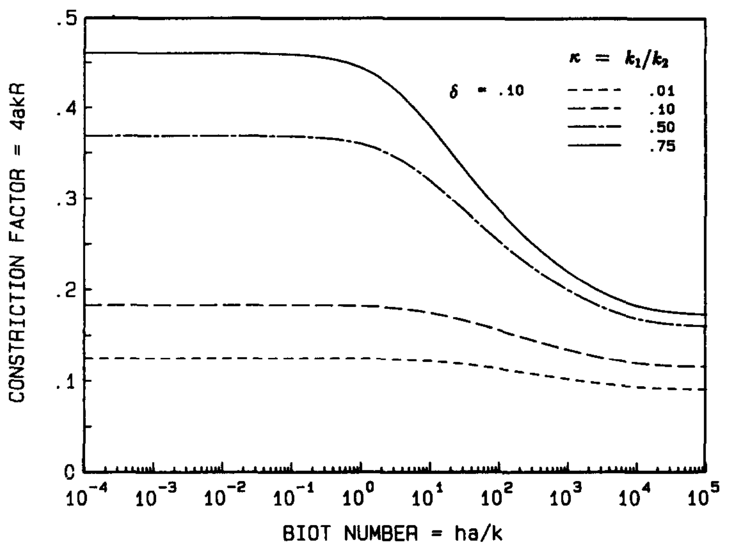

FIG. 7. Constriction factor vs Biot number; $\kappa < 1$, $\delta = 0.1$ external $T = 0$ boundary.

lar constriction resistance, suggesting the possible existence of another dimensionless quantity for the coated half-space contacts.

7. CONCLUSIONS

The results in this work provide the necessary asymptotic bounds for the more general layered problem of a convective external boundary with contact conductance. The effects of Biot number on the constriction resistance follow similar trends to those observed in Part 1. Further extensive results are also given in ref. [6] for single layer boards with various bottom surface conditions. We note too, that alternative integral operators may have been used to reduce the problems to Fredholm-type integral equations. However, these would then require suitable collocation procedures [7] for solution, and evaluation of kernel integrals at these points. It was found that the approach used here, whereby Fourier expansions reduced the integrals to a system of linear algebraic equations, was more concise and stable. The analogous kernel integrals were lumped in the influence R matrix, and efficient procedures for evaluating these were incorporated. Use was made of the theta-function theory [8] to compute accurately the complete elliptic integrals. Efficient series acceleration algorithms [9, 10] were also implemented for evaluating the various infinite series forms.

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RESISTANCE THERMIQUE DE CONSTRICTION AVEC DES CONDITIONS AUX LIMITES CONVECTIVES—2. CONTACTS LAMELLAIRES EN DEMI-ESPACE

Résumé—On considère l'analyse thermique axisymétrique d'un contact liée à un demi-espace. La méthode de transformation intégrale de Hankel est utilisée et des conditions aux limites convectives sont imposées sur la surface de contact. Dans chaque cas des développements de Fourier convenables réduisent le problème à la résolution d'équations intégro-différentielles semblables à celles étudiées dans la première partie pour des contacts de demi-espace. Des expressions compactes sont développées et la variation de la résistance thermique de constriction est montrée sous une forme adimensionnelle pour un large domaine du nombre de Biot, d'épaisseur de couche et du rapport de conductivité thermique.

DER THERMISCHE WIDERSTAND BEI KONVEKTIVEN RANDBEDINGUNGEN—2. KONTAKTE AN EINEM BESCHICHTETEN HALBRAUM

Zusammenfassung—Diese Arbeit beschäftigt sich mit der achsensymmetrischen thermischen Analyse des Kontakts in einer halbunendlichen einzelnen Schicht, die ideal an einem halbunendlichen Körper anliegt. Die Integraltransformationsmethode von Hankel wurde angewandt und konvektive Randbedingungen an der Kontaktoberfläche angesetzt. In allen Fällen reduzieren geeignete Fourier-Entwicklungen das Problem der Lösung der Integral-Differential-Gleichungen ähnlich denen in Teil 1 für Halbraum-Kontakte betrachteten. Kompakte Ausdrücke wurden entwickelt. Die Veränderung des thermischen Widerstandes wird in dimensionsloser Form für einen großen Bereich der Biot-Zahl, der Schichtdicke und der Wärmeleitfähigkeiten gezeigt.

ТЕРМИЧЕСКОЕ СОПРОТИВЛЕНИЕ ПРИ СЖАТИИ ДЛЯ КОНВЕКТИВНЫХ ГРАНИЧНЫХ УСЛОВИЙ—2. КОНТАКТЫ В СЛОИСТОМ ПОЛУПРОСТРАНСТВЕ

Аннотация—Методом интегрального преобразования Ханкеля проведен осесимметричный тепловой анализ контакта на полубесконечном единичном слое, идеально прилегающем к полупространству. На поверхность контакта налагаются конвективные граничные условия. В каждом рассматриваемом случае с помощью соответствующего разложения Фурье задача сводится к решению интегро-дифференциальных уравнений, аналогичных рассмотренным в первой части работы. Получены простые соотношения, описывающие изменение термического сопротивления при сжатии в безразмерном виде для широкого диапазона изменений числа Био, толщины слоя и отношения теплопроводностей.